National 5 Mathematics
Revision Notes

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Use this booklet to practise working independently like you will have to in the exam.

- Get in the habit of turning to this booklet to refresh your memory.
- If you have forgotten how to do a method, **examples** are given.
- If you have forgotten what a word means, use the **index** (back pages) to look it up.

As you get closer to the final exam, you should be aiming to use this booklet less and less.

This booklet is for:

- Students doing the National 5 Mathematics course.
- Students studying one or more of the National 5 Mathematics units: **Expressions and Formulae**, **Relationships** or **Applications.**

This booklet contains:

- The most important facts you need to memorise for National 5 Mathematics.
- Examples that take you through the most common **routine** questions in each topic.
- Definitions of the key words you need to know.

Use this booklet:

- To refresh your memory of the method you were taught in class when you are stuck on a homework question or a practice test question.
- To memorise key facts when revising for the exam.

<u>The key to revising for a maths exam is to do questions, not to read notes.</u> **As well as using this booklet, you should also:**

- Revise by working through exercises on topics you need more practice on – such as revision booklets, textbooks, websites, or other exercises suggested by your teacher.
- Work through practice tests.
- Ask your teacher when you come across a question you cannot answer.
- Use resources online (a QR code is on the last page).

Formula Sheet

Formulae that <u>are given</u> on the formula sheet in the exam (or in unit assessments)

Topic	Formula(e)	Page Reference
Volume of a Pyramid	$V=\frac{1}{3}Ah$	See page 41
Volume of a Cone	$V=\frac{1}{3}\pi r^2 h$	See page 42
Volume of a Sphere	$V=\frac{4}{3}\pi r^3$	See page 42
The Quadratic Formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	See page 65
Area of a Triangle	$A=\frac{1}{2}ab\sin C$	See page 90
Sine rule	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$	See page 91
Cosine rule	$a^2=b^2+c^2-2bc\cos A$ or $\cos A=\frac{b^2+c^2-a^2}{2bc}$	See page 93
Standard deviation	$\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$ or $\sqrt{\frac{\sum x^2-\left(\sum x\right)^2/n}{n-1}}$	See page 113

Formulae that are <u>not</u> given in the exam (or in unit assessments)

Topic	Formula(e)	Page Reference
Gradient	$m=\frac{y_2-y_1}{x_2-x_1}$	See page 37
Arc length	$\frac{\text{Angle}}{360}\pi d$	See page 39
Sector Area	$\frac{\text{Angle}}{360}\pi r^2$	See page 40
Volume of a prism	$V=Ah$	See page 41
Volume of a cylinder	$V=\pi r^2 h$	See page 41
Straight line	$y-b=m(x-a)$	See page 46
Discriminant	b^2-4ac	See page 66
Pythagoras' Theorem	$a^2+b^2=c^2$	See page 73
Right-angled Trigonometry	$\sin x^\circ=\frac{\text{Opp}}{\text{Hyp}}$ $\cos x^\circ=\frac{\text{Adj}}{\text{Hyp}}$ $\tan x^\circ=\frac{\text{Opp}}{\text{Adj}}$	See pages 77 and 82
Angles in regular polygons	Angle inside regular polygon (n sides) $=180-\frac{360}{n}$	See page 69
Similar Shapes	Scale Factor for Length $(s)=\frac{\text{Length in 'new' shape}}{\text{Length in 'old' shape}}$ Scale Factor for Area $=s^2$ Scale Factor for Volume $=s^3$	See page 70

Brackets and Factorising

Expanding Single Brackets

At National 4 level, you learnt to expand brackets (also referred to as "multiplying out brackets"). At that level, there would only have been a number in front of a bracket.

Example 1 – numbers in front of brackets

Expand the brackets and simplify: $2(7y+5)-4(2y-3)$

Solution

$2(7y+5)-4(2y-3)$

$=2\times 7y \quad +2\times 5 \quad +(-4)\times 2y \quad +(-4)\times(-3)$

$= \quad 14y \quad +10 \quad -8y \quad +12$ (expanding both brackets)

$=\underline{\underline{6y+22}}$ (collecting like terms)

At National 5 level, there may be letters in front of a bracket. There may also be *both* letters and numbers. The rule remains the same: multiply everything inside the bracket by the number(s) and/or letter(s) outside the bracket.

Examples 2 – letters in front of brackets

Expand the brackets:

(a) $m(m+5)$ **(b) $2a(3a+4b)$**

Solution

(a)

$m(m+5)$

$=m\times m \quad + m\times 5$

$= \quad \underline{m^2 \quad + \quad 5m}$

(b)

$2a(3a+4b)$

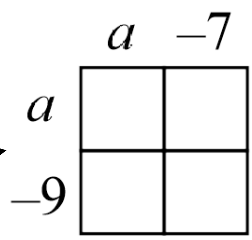
$=2a\times 3a+2a\times 4b$

$= \quad \underline{6a^2 \quad + \quad 8ab}$

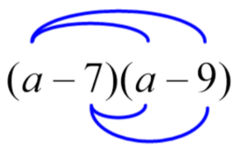
Expanding Double Brackets

To expand double brackets, you must multiply every term in the first bracket by *every* term in the second bracket. **You must always simplify your answers – <u>be very careful with negative signs</u>**.

A variety of methods are taught to ensure every term is multiplied. You are likely to use the one your teacher has shown you. Two of the most common methods below are illustrated for $(a-7)(a-9)$, which is the question from Example 1:



- Using a 2×2 grid (like the grid method for multiplication of numbers on page 10).
- Drawing lines between pairs of terms, using a method such as:
  - Drawing lines in order to create a 'moon' or 'face' shape.
  - Using 'FOIL' ("First, Outer, Inner, Last).



Key fact:
To factorise $ax^2 + bx + c$ (when there is a number in front of the x^2), we need one number to go in each bracket. The two numbers that we need:

- WILL multiply to make *c* but they WILL NOT add to make *b*

The key to this method is to experiment and check, although your teacher has probably taught you some 'smart' strategies for choosing numbers, which you should try to use!

Example 2a – where there is a number before x^2

Factorise $3x^2 + 11x + 6$.

Solution

Step 1: make a list of all possible pairs of numbers – **order matters**.
The two numbers in the bracket will multiply to give **+6**. So possibilities are:

+6 and +1, +1 and +6,
+2 and +3 +3 and +2.

(technically we should also consider –6 and –1; –3 and –2 etc. As these also multiply to give +6. However in this question we can ignore negative numbers as there are no negative signs)

Step 2: To make $3x^2$, we need $3x \times x$, so write $3x$ and x at the start of each bracket.

$$(3x \quad)(x \quad)$$

Step 3: experiment – try different pairs of numbers in the bracket. Multiply the brackets out to see if you can get $3x^2 + 11x + 6$.

- first you might try $(3x+6)(x+1)$. But this multiplies out to give $3x^2 + 9x + 6$, so this is NOT the answer.
- next you might try switching the '6' and '1' about to get $(3x+1)(x+6)$. But this multiplies out to give $3x^2 + 19x + 6$, so this is NOT the answer.
- next you might try $(3x+2)(x+3)$. This multiplies out to give $3x^2 + 11x + 6$, which is want we want, so this is the answer.

Answer: $(3x+2)(x+3)$

Step 4: check the answer by multiplying it back out to check.

There is an alternative method for factorising trinomials. The **advantages** of this method over the 'experiment and check' method are that it ought to work on the first attempt, that we don't need to consider the effect of changing the order of the numbers, and that it will work for *any* trinomial.

The **disadvantages** are that the algebra is more complicated, you need to be good at finding *all* factors of larger numbers, it requires a good understanding of the common factor method, and some people find this method harder to remember.

Algebraic Fractions

Simplifying Algebraic Fractions

You can simplify a fraction when there is a common factor (number *or* letter) on the top *and* on the bottom.

e.g. $\frac{9xy^2}{18x^2y} = \frac{\not{9}\,\not{x}\,y^{\not{2}}}{\not{18}\,x^{\not{2}}\,\not{y}} = \frac{y}{2x}$

A factor may also be an entire bracket. In National 5 assessments, questions will be of this form. In this case you can cancel the entire bracket if the entire bracket is identical on the top and the bottom. If you cannot cancel the entire bracket, then you cannot cancel anything inside the bracket.

Example 1

Simplify: **(a)** $\frac{(a+5)(a-1)}{(a-1)(a+2)}$ **(b)** $\frac{(x+3)(x-2)}{(x-2)^2}$

Solution

(a)

$$\frac{(a+5)(a-1)}{(a-1)(a+2)}$$
$$= \frac{(a+5)\cancel{(a-1)}}{\cancel{(a-1)}(a+2)}$$
$$= \frac{(a+5)}{(a+2)}$$

(b)

$$\frac{(x+3)(x-2)}{(x-2)^2}$$
$$= \frac{(x+3)\cancel{(x-2)}}{(x-2)\cancel{(x-2)}}$$
$$= \frac{(x+3)}{(x-2)}$$

Important: neither of these answers can be cancelled any further as the remaining brackets are different. You cannot cancel the *a* or the *x*, as they are not common factors.

In exam questions, if you are asked to simplify, you will need to factorise first and then cancel brackets.

Example 2

Simplify $\frac{v^2-1}{v-1}$.

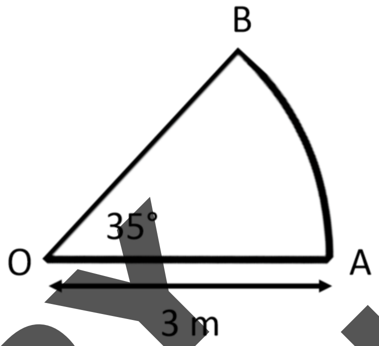
Solution

v^2-1 is a difference of two squares (see page 26), which factorises to be $(v+1)(v-1)$:

$$\frac{v^2-1}{v-1} = \frac{(v+1)\cancel{(v-1)}}{\cancel{(v-1)}}$$
$$= \frac{v+1}{1}$$
$$= v+1$$

Example 2 – Sector area

Calculate the area of sector OAB shown in this diagram.



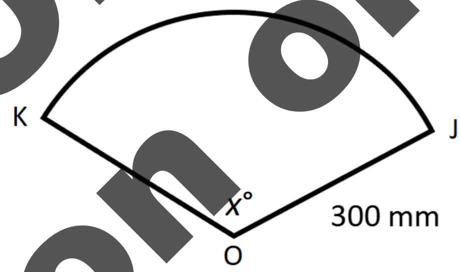
Solution

$$A = \frac{35}{360}\pi r^2$$
$$= 35 \div 360 \times \pi \times 3^2$$
$$= 2{\cdot}74889357...$$
$$= \underline{2{\cdot}75 \text{ m}^2 \text{ (2 d.p.)}}$$

Note: units for sector area must always be squared units.

Example 3 – Arc length backwards

The diagram shows a sector OJK of a circle with radius 300 mm.
The length of arc JK is 550 mm.
Calculate the size of angle OJK (marked *x* in the diagram).



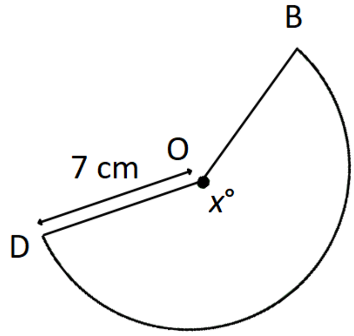
Solution

We use the normal formula for arc length and then go backwards to calculate the angle by doing inverse operations to both sides.

$$\frac{x}{360}\pi d = \text{Arc length}$$
$$\underset{\times 360}{x \div 360} \times \underset{\div \pi}{\pi} \times \underset{\div 600}{600} = \underset{\times 360 \div \pi \div 600}{550}$$
$$x = 550 \div 600 \div \pi \times 360$$
$$x = 105.04...$$
$$= \underline{105.0^\circ \text{ (1 d.p.)}}$$

Example 4 – Sector area backwards

The diagram shows a sector ODB of a circle with radius 7 cm.
The area of sector ODB is 83·38 cm².
Calculate the size of angle BOD (marked *x* in the diagram).



Solution

We use the normal formula for sector area and then go backwards to calculate the angle by doing inverse operations to both sides.

$$\frac{x}{360}\pi r^2 = \text{Sector Area}$$
$$\underset{\times 360}{x \div 360} \times \underset{\div \pi}{\pi} \times \underset{\div 7^2}{7^2} = \underset{\times 360 \div \pi \div 7^2}{83{\cdot}38}$$
$$x = 83{\cdot}38 \div 7^2 \div \pi \times 360$$
$$x = 194.992...$$
$$= \underline{195.0^\circ \text{ (1 d.p.)}}$$

Equations and Inequations

At National 5 level, you need to be able solve more complex equations and inequations. The method used in these notes is to *do the same operation to both sides of the equation*. You must use a method to get the answer – ***if you just write the answer down (even if you think it is obvious) you will get no marks.*** You should always double check your final answer so that you know it is correct.

Equations and Inequations with Letters on Both Sides

At National 4 level, you learnt to solve an equation that has letters on both sides. At National 5 level you need to be able to use this method and to extend it to inequations and more difficult equations. The basic method is as follows:

- ***[Before you start*** *(optional) – write the "invisible plus signs" in, in front of anything that does not have a sign in front of it already, to remind you it is positive.]*
- Step one: use inverse operations to add or subtract from both side in order to remove terms. Our aim is to end up with all the terms involving a letter ('variables') on one side of the equals sign (usually the left, but not always) and all the term that are just numbers ('constants') on the other side (usually the right).
- Step two: simplify both sides.
- Step three: divide to solve the resulting equation.
- Final step: double check your answer is correct by substituting it back in to both sides of the original equation and checking both sides give the same answer.

Example 1 – basic method from National 4

Solve, algebraically, the equation $2a+5=15-3a$.

Solution

Optional starting step*: write in "invisible plus signs" in front of anything that does not already have a sign*

Step one: remove the '–3*a*' from the right-hand side by adding 3*a* to both sides. Remove the '+5' from the left-hand side by subtracting 5 from both sides.

Step two: simplify both sides.

Step three: divide to get the final answer.

$$2a+5=15-3a$$

$$+2a+5=+15-3a$$

$$(+3a,\ -5 \quad -5,\ +3a)$$

$$+2a+3a=+15-5$$

$$5a=10 \quad (\div 5 \ \ \div 5)$$

$$a=\frac{10}{5}$$

$$a=2$$

Final step: double check, by substituting *a* = 2 into both sides of the original equation:

- The left-hand side is 2*a* + 5. If we substitute *a* = 2, we get 2 × 2 + 5, which is 9.
- The left-hand side is 15 – 3*a*. If we substitute *a* = 2, we get 15 – 3 × 2, which equals 9. These are the same, so our answer must be correct.

Inequalities (also known as Inequations) are solved with exactly the same method as equations, except that there is not a '=' sign in the middle, there will be another type of sign.

Quadratic Functions

Definition: a quadratic function is one that contains a squared term and no higher powers.
e.g. $3x^2$, x^2-4 and x^2+5x+1 are all quadratic functions, but x^5 and x^2+x^3 are not.

Function Notation

You need to be familiar with function notation.

Example 1

Three functions are defined by $f(x)=x^2+4$, $g(x)=5-x$ and $h(t)=4+7t$.
Calculate:
(a) $f(7)$ (b) $h(3)$ (c) $g(-2)$

Solutions

$f(x)=x^2+4$
$f(7)=7^2+4$
$=53$

$h(t)=4+7t$
$h(3)=4+7\times 3$
$=25$

$g(x)=5-x$
$g(-2)=5-(-2)$
$=7$

Example 2

A function is defined as $f(x)=6x-3$. Given that $f(a)=42$, calculate a.

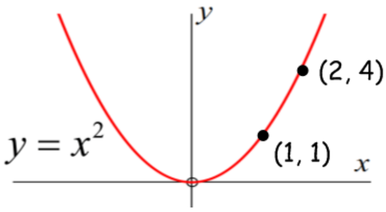
Solution

$f(a)=42$
$6a-3=42$
$6a=45$
$a=\frac{45}{6}=7\cdot 5$

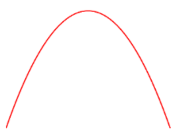
Graphs of Quadratic Functions

Definition: the graph of a quadratic function is in a shape known as a **parabola**.

The graph on the right is the basic graph of $y=x^2$.
You must know its shape and that it goes through (0, 0).



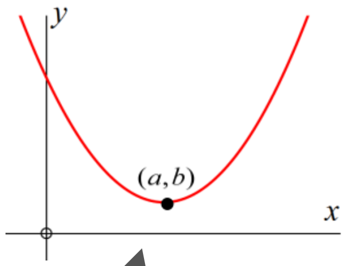
If x^2 is positive, the graph is "happy" (it has a **minimum** turning point).



If x^2 is negative, then the graph is "unhappy" (it has a **maximum** turning point).

The graph of $y=(x\pm a)^2+b$ is still a parabola, but it has been moved so that its minimum point is no longer at (0, 0):

- The number inside the bracket (*a*) tells us how far the graph has been moved **left or right**. If *a* is positive, the graph moved to the left. If it is negative, it moved to the right.
- The number outside the bracket (*b*) tells us how far the graph has been moved **up or down**. If *b* is positive, the graph moved upwards. If it is negative, it moved downwards.



Key facts:

- $y=(x-a)^2+b$ is a "happy" parabola. Its minimum turning point is (*a*, *b*).
- $y=-(x-a)^2+b$ is an "unhappy" parabola. Its maximum turning point is (*a*, *b*).
- The **axis of symmetry** of $y=(x-a)^2+b$ or $y=-(x-a)^2+b$ has the equation *x* = *a*.

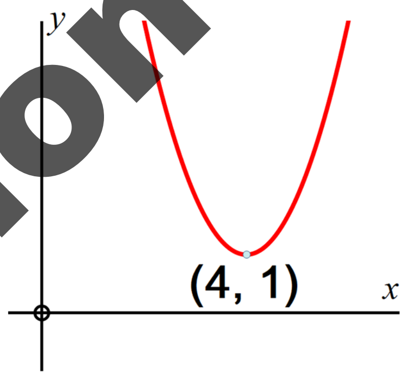
For example, the graph with equation $y=(x-5)^2+4$ is a 'happy' parabola that has been moved 5 to the right and 4 up. Its minimum turning point is (5, 4) and it has axis of symmetry *x* = 5.

Example 1

The graph shown has an equation of the form $y=(x+a)^2+b$. It has minimum turning point (4, 1) .

a) State the equation of the graph.

b) Write down the equation of the axis of symmetry.



Solution

a) The minimum point is (4, 1). This tells us that the graph has been moved 4 to the right (*a* = –4) and 1 up (*b* = 1).
Therefore, its equation is $\underline{y=(x-4)^2+1}$.

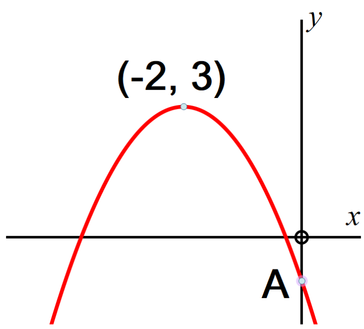
b) The equation of the axis of symmetry is $\underline{x = 4}$.

Example 2

The graph shown has an equation of the form $y=b-(x+a)^2$.
It has maximum turning point (–2, 3) .

a) State the equation of the graph.

b) A is the point (0, *c*). Find the value of *c*.



Solution

a) The maximum point is (–2, 3). This tells us that the graph has been moved 2 to the left (*a* = 2) and 3 up (*b* = 3).
Therefore, its equation is $\underline{y=3-(x+2)^2}$.

Example 5 – right-hand side is not equal to zero

Solve the equation $x^2 - 2x - 10 = 5$.

Solution

Step 1 – check that the equation has '0' on the right-hand side.

*It **does not**, so we need to rearrange by subtracting 5 from both sides.*

$$x^2 - 2x - 10 = 5 \quad (-5,\ -5)$$
$$x^2 - 2x - 10 - 5 = 0$$
$$x^2 - 2x - 15 = 0$$

Step 2 – factorise the rearranged expression.

$$x^2 - 2x - 15 = 0$$
$$(x+3)(x-5) = 0$$

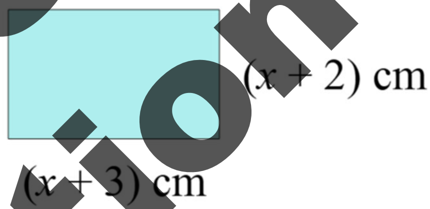
Step 3 – split up into two separate equations and solve each equation separately.

$$x + 3 = 0 \qquad x - 5 = 0$$
$$x = -3, \qquad x = 5$$

Example 6 – in a real-life context

The area of the rectangle shown is 42 square centimetres.

Form a quadratic equation and calculate, algebraically, the value of *x*.



Solution

Step 1 – use the formula for area of a rectangle.

$$\text{Area} = \text{Length} \times \text{Breadth}$$
$$A = (x+3)(x+2)$$

Step 2 – substitute for the area.
We are told the area is 42 cm² so we substitute $A = 42$.

$$42 = (x+3)(x+2)$$

Step 3 – expand the brackets and rearrange to get '= 0'.

$$(x+3)(x+2) = 42$$
$$x^2 + 5x + 6 = 42 \quad (-42,\ -42) \qquad \text{(expanding brackets)}$$
$$x^2 + 5x + 6 - 42 = 0 \qquad \text{(subtracting 42 from each side)}$$
$$x^2 + 5x - 36 = 0 \qquad \text{(collecting like terms)}$$

Step 4 – factorise the rearranged expression.

$$(x+9)(x-4) = 0$$

Step 5 – split up into two separate equations and solve each equation separately.

$$x + 9 = 0 \qquad x - 4 = 0$$
$$x = -9, \qquad x = 4$$

Step 6 – check whether each solution makes sense in the given real-life context. If it does not, reject it.
An answer of $x = 4$ makes sense. However an answer of $x = -9$ makes no sense as it would lead to negative lengths. So we reject $x = -9$ and our final answer is $x = 4$.

The Quadratic Formula

Formula. This formula is <u>given</u> on the National 5 Mathematics exam paper.

The roots of $ax^2+bx+c=0$ are given by: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

The quadratic formula can be used to solve *any* quadratic equation. We usually use it when you can't factorise the expression. Answers usually require rounding, and so the question may include the instruction "**give your answers correct to 2** (or 1) **decimal places**" etc. Remember you should always state your unrounded answer before rounding (see page 7).

Important – you <u>must</u> rearrange the equation so that is has = 0 on the right-hand side. If you do not do this, you will risk losing <u>all</u> of the marks.

In these questions, '*a*' is the number in front of x^2, *b* is the number in front of *x*, and *c* is the constant term. For example, for $4x^2-5x+9=0$:

$a = 4$, $b = -5$, $c = 9$

If the number under the square root sign works out to be negative, then you will not be able to complete the formula. This means either that:

- You have made a mistake with negative numbers and need to check your working (realistically this is the most likely thing that would have happened in an exam).
- Or the equation has no real roots (happens a lot in real life, but less likely in an exam).

<u>Example 1</u>

Solve the equation $3x^2+2x-6=0$.
Give your answers correct to one decimal place.

Solution

<u>Step 1:</u> check that the equation has '= 0' on the RHS. It does, so we can proceed.

<u>Step 2:</u> state the values of *a*, *b* and *c*: $a = 3$, $b = 2$, $c = -6$

<u>Step 3:</u> calculate b^2-4ac (the discriminant, which will be square rooted in Step 4).

$b^2-4ac=2^2-4\times3\times(-6)=4-(-72)=\underline{\underline{76}}$

<u>Step 4:</u> substitute into the formula and solve.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-2\pm\sqrt{76}}{6}$$

$$x=\frac{(-2+\sqrt{76})}{6}$$
$=1\cdot119...$
$=1\cdot1$ (1 d.p.)

$$x=\frac{(-2-\sqrt{76})}{6}$$
$=-1\cdot786...$
$=-1\cdot8$ (1 d.p.)

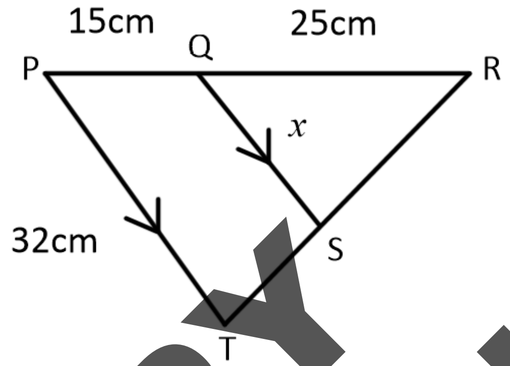
Essential to remember to include brackets when typing into the calculator!

Example 4 – similar triangles

In the diagram, triangles PRT and QRS are mathematically similar.

PQ = 15 cm, QR = 25 cm and PT = 32 cm.

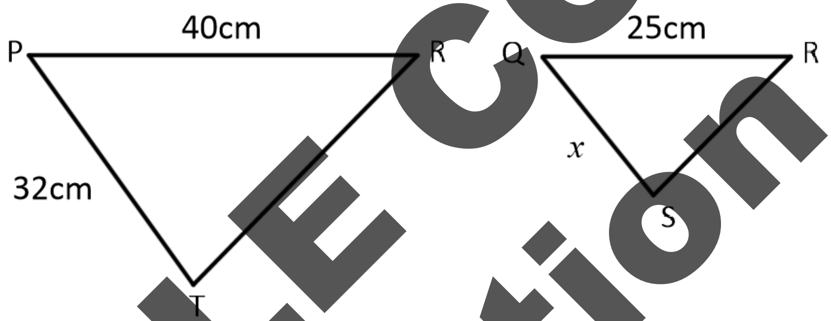
Calculate the length, *x*, of QS.



Solution

Firstly, notice that PR = 15 + 25 = 40 cm.

Next, it will be less confusing if we redraw the diagram so that the two triangles can be seen side by side:



Now, using the method from Example 1 and Example 2:

Step one: calculate the scale factor for lengths.

$$\text{Scale Factor for Lengths} = \frac{25}{40} = 0\cdot625$$

Step two: calculate the length. The question is not about area or volume, so we do not need to square or cube the scale factor:

Length = $32 \times 0\cdot625 = \underline{\underline{20\ \text{cm}}}$

Converse of Pythagoras

Pythagoras' Theorem says that if a triangle is right-angled, then $a^2 + b^2 = c^2$.
It can also be used backwards: **if $a^2 + b^2 = c^2$, then the triangle must be right-angled**.
This rule is called the **Converse of Pythagoras**, and can be used in a triangle (when all three sides are known) to check whether or not an angle is a right-angle.

The procedure is:

1. Calculate the square of the longest side.
2. Do a separate calculation in which you calculate the sum of squares of the two shorter sides.
3. Write a conclusion:
    a. If the two answers are equal, then conclude that the angle is a right angle.
    b. If the two answers are not equal, then conclude that the angle is not a right angle.

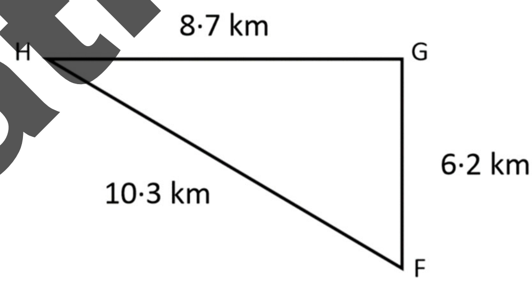
Marks for writing conclusions usually expect precision and detail. In past mark schemes, a conclusion has typically required three criteria to gain a mark:

1. An answer to the question asked (e.g. 'yes' or 'no').
2. A numerical comparison of the two calculations. e.g. 577·2 = 577·2 or 91 ≠ 88.
3. A clear statement saying either that the angle 'is a right angle' or 'is not a right angle' (or equivalent wording such as 'is 90°').

Example

The diagram shows three towns F, G and H.
The distances between towns are shown.

G is due East of H.
Determine whether G is due North of F.



Solution

Do not begin by writing $8\cdot7^2 + 6\cdot2^2 = 10\cdot3^2$.
This is not correct as we are checking whether it's true and so cannot state it yet. You will lose at least one mark.

Step one: calculate the square of the longest side:

$$c^2 = 10\cdot3^2 = 106\cdot09$$

Step two: calculate the sum of squares of the two shorter sides:

$$a^2 + b^2 = 8\cdot7^2 + 6\cdot2^2$$
$$= 114\cdot13$$

Step three: write a conclusion containing all three aspects listed above:

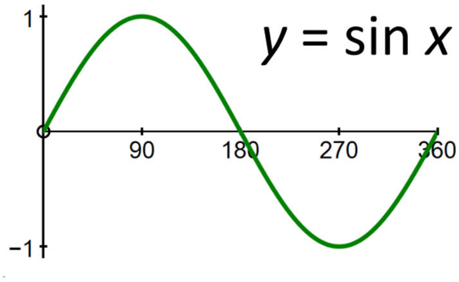
The following sentences would be likely to be suitable (you would only have to write one of these):

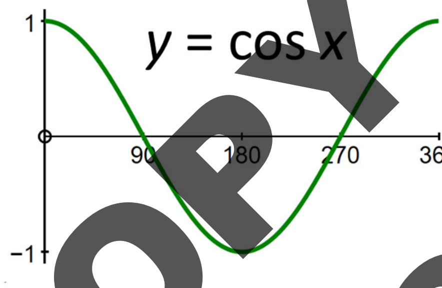
- $114\cdot13 \neq 106\cdot09$ so the angle is not 90° so no.
- No because $114\cdot13 \neq 106\cdot09$, so angle HGF is not a right-angle.
- No. Not right-angled. $8\cdot7^2 + 6\cdot2^2 \neq 10\cdot3^2$.

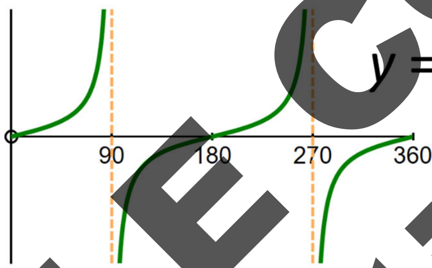
Trigonometric Graphs and Equations

Graphs of Trigonometric Functions (sin, cos and tan)

You must know what the basic graphs of sin *x*, cos *x* and tan *x* (trigonometric functions) look like between 0° and 360°.







In the graph of y = tan x pictured above the dotted lines are not part of the graph but they indicate where the graph extends infinitely high/low either side of the line. The graph never quite touches the dotted line.

All of the graphs can be extended before 0° and beyond 360°. When extended, the graphs repeat the exact same shape infinitely many times. They are called periodic functions.

Definition:
The **frequency** of a trigonometric function is how many times the basic graph shape repeats in 360°.

Definition: the **amplitude** is a measure of the 'height' of a sin or cos graph:
- the graphs above (with maximum 1 and minimum –1) both have an amplitude of 1.
- a sin or cos graph with a maximum of 8 and a minimum of –8 would have amplitude 8.

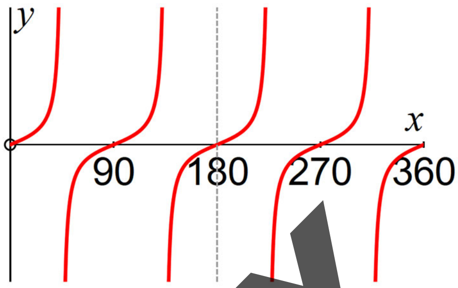
Definition: the **period** of a graph describes how many degrees it takes to make one complete cycle of the basic graph.

$$\text{Period} = \frac{360°}{\text{Frequency}}$$

- In the equation of a sin, cos or tan graph, the frequency is the number multiplying *x*. It is represented by *b* in the equations $y = \sin bx$, $y = \cos bx$ or $y = \tan bx$.
- In the equation of a sin or cos graph, the amplitude is the number multiplying sin or cos. It is represented by *a* in the equations $y = a\sin x$ or $y = a\cos x$.

Example 4

The graph shown has an equation of the form $y = \tan bx$. State the value of b.



Solution

The basic graph of tan repeats twice (compare to the diagram on page 79). Therefore the frequency is 2.

Answer: b = 2 so the graph is $y = \tan 2x$.

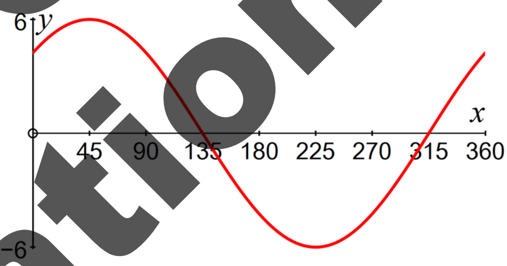
Definition: the **phase angle** is the amount a graph has been translated (moved) to the left. In the equation of a sin, cos or tan graph, the phase angle is the number added to x in the brackets.

Equation	Frequency	Amplitude	Phase Angle	Moved
$y = 2\cos(x+45)°$	1	2	45°	45° left
$y = \sin(x-30)°$	1	1	(–30)°	30° right

Example 5

The graph shown has an equation of the form $y = a\sin(x+b)°$.

State the values of a and b.



Solution

The maximum and minimum are 6 and –6, so the amplitude is 6, therefore a = 6.
The graph has been translated 45° to the left, so b = 45°. Answer: a = 6 and b = 45 so the graph is $y = 6\sin(x-45)°$.

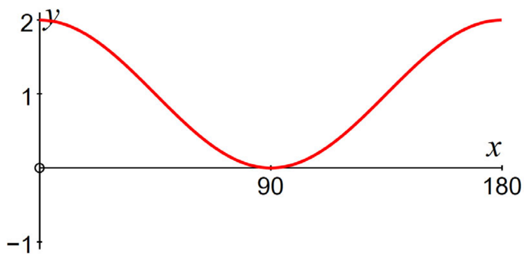
Definition: a **vertical translation** is the amount a graph has been moved up (down if the number is negative). In the equation of a sin, cos or tan graph, the vertical translation is the number added at the end of an equation.

Equation	Amplitude	Vertical translation	Moved
$y = 2\cos x + 3$	2	3	3 up
$y = 5\sin 2x - 1$	5	–1	1 down

Example 6

The graph shown has an equation of the form $y = \cos bx + c$.

State the values of b and c.



Solution

b is the frequency. The graph repeats once in 180°, meaning it will repeat twice in 360° therefore b = 2.
A cos graph would normally have y values between 1 and –1. This graph goes between 2 and 0. This means it has been moved up 1. Therefore c = 1.
Answer: the graph is $y = \cos 2x + 1$.

Applications Unit

Trigonometry in Scalene Triangles

Area of a Triangle

To find the area of any triangle you need the length of two sides and the size of the angle in between them.

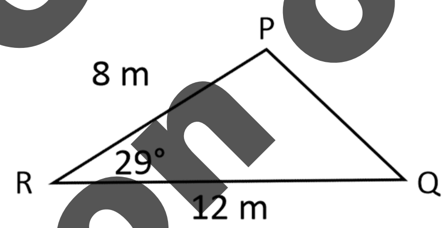
Formula. This formula is given on the National 5 Mathematics exam paper.

Area of a Triangle: $A = \frac{1}{2}ab\sin C$

where *a* and *b* are the lengths, and *C* is the angle in between.

Example 1 – calculator allowed

Calculate the area of triangle PQR.
Give your answer correct to 1 decimal place.



Solution

In the formula, *a* and *b* are the lengths, so we use $a = 8$, and $b = 12$.

In the formula *C* is the angle between the two lengths, so $C = 29°$:

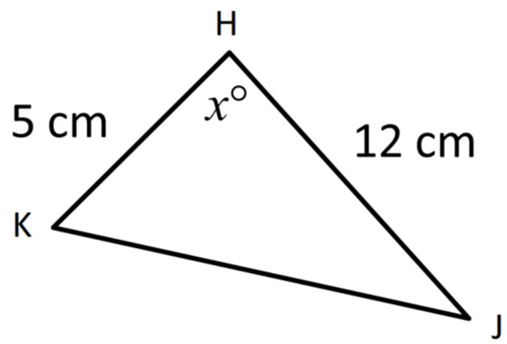
$$A = \frac{1}{2}ab\sin C$$
$$A = 8 \times 12 \times \sin 29° \div 2$$
$$A = 23{\cdot}27086177...$$
$$\underline{A = 23{\cdot}3 \text{ m}^2} \text{ (1 d.p.)}$$

Example 2 – non-calculator

In triangle HJK:

- **HK = 5 centimetres**
- **HJ = 12 centimetres**
- **sin H =** $\frac{3}{10}$

Calculate the area of triangle HJK.



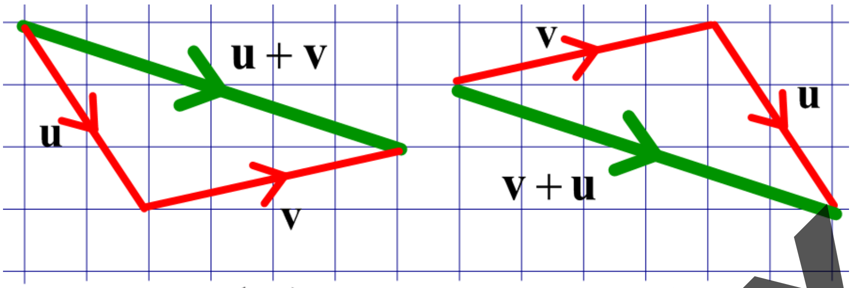
Solution

In the formula, *a* and *b* are the lengths, so we use $a = 5$, and $b = 12$.

In the formula *C* is the angle between the two lengths, so $\sin C = \frac{3}{10}$:

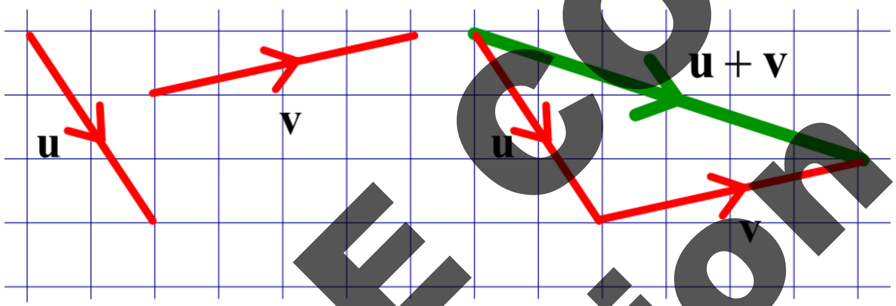
$$A = \frac{1}{2}ab\sin C$$
$$A = (5 \times 12) \times \frac{3}{10} \div 2$$
$$A = 60 \div 10 \times 3 \div 2$$
$$\underline{A = 9 \text{ cm}^2}$$

2. **In a diagram** by joining them 'nose to tail'.



$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}6\\-2\end{pmatrix} \qquad \mathbf{v}+\mathbf{u}=\begin{pmatrix}6\\-2\end{pmatrix}$$

$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$, i.e. the order in which vectors are added does not matter:



$$\mathbf{u}=\begin{pmatrix}2\\-3\end{pmatrix} \quad \mathbf{v}=\begin{pmatrix}4\\1\end{pmatrix} \quad \mathbf{u}+\mathbf{v}=\begin{pmatrix}6\\-2\end{pmatrix}$$

We can also **subtract** vectors by adding the negative vector. $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

In real life, resultant vectors can be used to work out what the combined effect of more than one force pulling on an object will be.

Example 1 – from components

Three forces act on an object.
The three forces are represented by the vectors a, b and c, where:

$$\mathbf{a}=\begin{pmatrix}-1\\3\\2\end{pmatrix} \quad \mathbf{b}=\begin{pmatrix}0\\-5\\6\end{pmatrix} \quad \mathbf{c}=\begin{pmatrix}4\\0\\2\end{pmatrix}$$

Calculate the resultant force. Express your answer in component form.

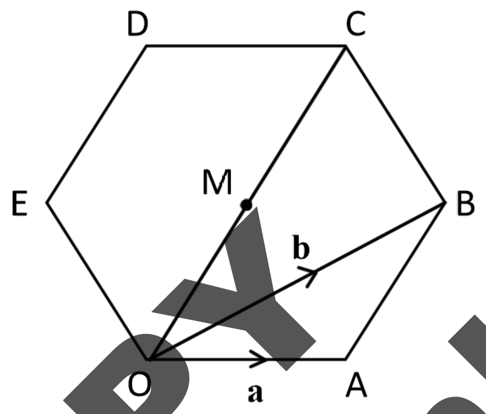
Solution

The resultant force is given by $\mathbf{a}+\mathbf{b}+\mathbf{c}=\begin{pmatrix}(-1)+0+4\\3+(-5)+0\\2+6+2\end{pmatrix}=\begin{pmatrix}3\\-2\\10\end{pmatrix}$.

Example – vector pathways (adapted from SQA specimen paper)

In the diagram, OABCDE is a regular hexagon with centre M. Vectors a and b are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

Express $\overrightarrow{AB}$ and $\overrightarrow{OC}$ in terms of a and b.



Solution

For $\overrightarrow{AB}$:

Step one: identify a path from A to B.

One possible path is $\overrightarrow{AO}$, $\overrightarrow{OB}$.

Step two: express each part of the path in terms of a known vector:

$\overrightarrow{AO} =$ backwards along $\mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$

Therefore $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$ (or $\mathbf{b} - \mathbf{a}$).

For $\overrightarrow{OC}$:

Step one: identify a path from O to C.

One possible path is $\overrightarrow{OM}$, $\overrightarrow{MC}$.

Step two: express each part of the path in terms of a known vector:

$\overrightarrow{OM}$ and $\overrightarrow{MC}$ are both the same as $\overrightarrow{AB}$ which is $\mathbf{b} - \mathbf{a}$ (from part (a))

Therefore $\overrightarrow{OC} = 2(\mathbf{b} - \mathbf{a})$ or $2\mathbf{b} - 2\mathbf{a}$.

Magnitude

The **magnitude** of a vector is the length of a vector. The magnitude of the vector **a** is written using two vertical lines, $|\mathbf{a}|$.

The magnitude of a vector is found using a version of Pythagoras' Theorem. There is also a three-dimensional equivalent of Pythagoras' theorem that can be used to find the magnitude of a 3-d vector when its components are known:

Formulae. These formulae are not given on the National 5 Mathematics exam paper.

The magnitude of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.

The magnitude of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$